PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Linearized Supersonic Aerofoil Theory. Part II

The Royal Society

Phil. Trans. R. Soc. Lond. A 1947 **240**, 356-373
doi: 10.1098/rsta.1947.0006

PART II

The operational approach to linearized supersonic aerofoil problems is further developed. In particular, the method is extended to give a general treatment of the drag on swept-back wings at zero incidence. Problems involving the lift on swept-back wings are also considered, and a recurrence method developed for obtaining the lift on a trapezoidal wing with tips swept back beyond the Mach angle. As a particular case this latter method is applied to the arrow-head wing.

Introduction

In Part I the author has applied operational methods to a variety of linearized supersonic aerofoil problems. The treatment so far, however, has been, in the main, limited to aerofoils with tips parallel to the undisturbed stream. It is now proposed to apply these methods to the treatment of a variety of 'swept-back' wing problems, the angle of sweep back being no longer restricted to be less than the Mach angle. Two main methods were developed in Part I, which may briefly be characterized as the Fourier integral, and the Green's function methods. The first of these was particularly suitable for dealing with aerofoils at zero incidence, for which, indeed, it provides a general solution. The second was employed for problems with lift. We shall begin by applying the first method to investigate the variation of wave drag on a swept-back wing at zero incidence. The second method will then be modified to deal with various lift problems.

In general, we deal with a supersonic stream, velocity U at infinity with undisturbed direction along the z-axis, impinging on an aerofoil of infinitesimal thickness lying, to the first order, in the plane $y = 0$. If ϕ denotes the velocity potential of the dimensionless flow disturbance, i.e. such that the disturbance velocity $\mathbf{q} = (u, v, w) = U \operatorname{grad} \phi$, then it is known that ϕ satisfies the equation

$$\nabla^2\phi = M^2 \frac{\partial^2 \phi}{\partial z^2},$$

where M is the Mach number of the incident flow. This equation is generally used in the form

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \alpha^2 \frac{\partial^2\phi}{\partial z^2}, \quad \alpha^2 = M^2 - 1. \tag{1}$$

Straight-edge aerofoils at zero incidence

We consider first the case of symmetrical aerofoils at zero incidence. It will be possible to build up a variety of straight-edge aerofoil cases by considering the simple problem of a pointed triangular-shaped aerofoil, as shown in figure 10, for which both the upper and lower surfaces have a constant slope ϵ, say, along the z-direction, so that they may be represented by the planes $y = \pm(z - mx)\,\epsilon$, where $m = \tan\tau$. Physically we shall have two very different states according as τ, the angle of sweep-back, is greater or less than $(\frac{1}{2}\pi - \mu)$, where μ is the Mach angle $(= \tan^{-1} 1/\alpha)$. Suppose, first, $\tau < \frac{1}{2}\pi - \mu$, then the leading edge OA (figure 10) is ahead of the Mach line OM, and within the triangle MOA conditions are as for an aerofoil of infinitely extended span. The conditions for such an infinitely extended aerofoil are easily obtained from two-dimensional theory. We find

$$\frac{\partial\phi}{\partial z} = -\frac{\epsilon}{\sqrt{(\alpha^2 - m^2)}} = -\frac{\epsilon\cos\tau}{\sqrt{(M^2\cos^2\tau - 1)}},$$

and the excess pressure $$\Delta p_0 = \rho u^2 \frac{\epsilon \cos \tau}{\sqrt{(M^2 \cos^2 \tau - 1)}}.$$

On the other hand, if $\tau > (\frac{1}{2}\pi - \mu)$, then the corresponding two-dimensional problem, in fact, becomes a subsonic one. For our three-dimensional problem, however, there is still a wave drag, as is physically evident from the hydrodynamic analogue of the long gravity waves produced by a plank of constantly increasing length, the two surfaces of which are pushed apart with constant velocity. It will be our object to investigate how the wave drag varies as the angle τ is increased.

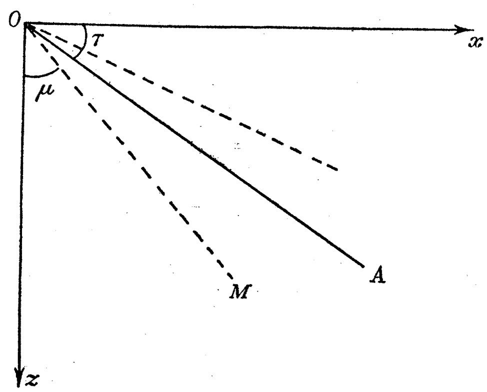

FIGURE 10

For the formal solution of the problem we introduce the Laplace transform

$$\bar{\phi}(x, y) = \int_0^\infty e^{-pz} \phi(x, y, z)\, dz.$$

This satisfies the equation $$\frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial y^2} = \alpha^2 p^2 \bar{\phi}, \tag{2}$$

and the boundary condition $$\left(\frac{\partial \bar{\phi}}{\partial y}\right)_{y=0} = 0 \quad (x<0),$$

$$\frac{\partial \bar{\phi}}{\partial y} \to \pm \frac{\epsilon}{p} e^{-mxp}, \quad \text{as } y \to \pm 0,\ x>0. \tag{3}$$

For $y>0$, we suppose the general solution of (2) written in the form

$$\phi = \int_{-\infty}^{\infty} a(q)\, e^{i\alpha qx}\, e^{-\alpha\sqrt{(p^2+q^2)}\,y}\, dq.$$

Using the boundary conditions (3), we find

$$a(q) = -\frac{\epsilon}{2\pi p\sqrt{(p^2+q^2)}} \int_0^\infty e^{-i\alpha q\xi}\, e^{-mp\xi}\, d\xi. \tag{4}$$

Thus $$\bar{\phi} = -\frac{\epsilon}{2\pi p} \int_{-\infty}^{\infty} \frac{e^{\alpha\{iqx - \sqrt{(p^2+q^2)}\,y\}}}{\sqrt{(p^2+q^2)}}\, dq \int_0^\infty e^{-\xi(mp+i\alpha q)}\, d\xi.$$

We are principally interested in the value of $\partial\phi/\partial z$, and for this we find, since the transform of

$$\frac{e^{-mp\xi-\alpha y\sqrt{(p^2+q^2)}}}{\sqrt{(p^2+q^2)}} \quad \text{is} \quad \begin{cases} 0, & z<\alpha y+m\xi \\ J_0[q\sqrt{\{(z-m\xi)^2-\alpha^2y^2\}}], & z>\alpha y+m\xi, \end{cases} \tag{5}$$

$$\frac{\partial\phi}{\partial z}=-\frac{\epsilon}{2\pi}\int_0^{z-\alpha y}d\xi\int_{-\infty}^{\infty}e^{i\alpha q(x-\xi)}J_0[q\sqrt{\{(z-m\xi)^2-\alpha^2y^2\}}]\,dq.$$

Thus
$$\frac{\partial\phi}{\partial z}=-\frac{\epsilon}{\pi}\int_0^{\xi_1}\frac{d\xi}{\sqrt{[(z-m\xi)^2-\alpha^2\{(x-\xi)^2+y^2\}]}}, \tag{6}$$

where the upper limit ξ_1 must be less than $(z-\alpha y)/m$, and throughout $(0,\xi_1)$,

$$(z-m\xi)^2>\alpha^2\{(x-\xi)^2+y^2\}.$$

The integral is easily evaluated for the case $y=0$. The results are listed below.

(*a*) $\alpha>m(\tau<(\tfrac{1}{2}\pi-\mu))$:

$$z>mx, \quad \text{but } <\alpha x, \quad \frac{\partial\phi}{\partial z}=-\frac{\epsilon}{\sqrt{(\alpha^2-m^2)}},$$

$$z>\alpha x, \quad \frac{\partial\phi}{\partial z}=-\frac{2\epsilon}{\pi\sqrt{(\alpha^2-m^2)}}\cos^{-1}\sqrt{\left\{\frac{(z-\alpha x)(\alpha+m)}{2\alpha(z-mx)}\right\}}$$

$$=-\frac{\epsilon}{\pi\sqrt{(\alpha^2-m^2)}}\cos^{-1}\left\{\frac{mz-\alpha^2x}{\alpha(z-mx)}\right\}. \tag{7}$$

Introducing $\sigma=\alpha x/z$, so that within the Mach triangle from O, $-1<\sigma<1$, we have in place of (7)

$$\frac{\partial\phi}{\partial z}=-\frac{\epsilon}{\pi\sqrt{(\alpha^2-m^2)}}\cos^{-1}\left(\frac{m-\alpha\sigma}{\alpha-m\sigma}\right). \tag{8}$$

As is to be expected $\partial\phi/\partial z$ tends to the two-dimensional value as σ approaches 1, and to zero as σ approaches -1. Equally it may be verified from (6) that on the Mach cone

$$(z^2=\alpha^2(x^2+y^2)),$$

$$\frac{\partial\phi}{\partial z}=-\frac{\epsilon}{\sqrt{(\alpha^2-m^2)}} \quad (0<\theta<\theta_1),$$
$$=0 \quad (\theta_1<\theta<\pi),$$

where $\theta=\tan^{-1}y/x$, and $\cos\theta_1=m/\alpha$.

(*b*) $\alpha<m(\tau>(\tfrac{1}{2}\pi-\mu))$:

$$z>\alpha|x|, \quad \frac{\partial\phi}{\partial z}=-\frac{\epsilon}{\pi\sqrt{(m^2-\alpha^2)}}\cosh^{-1}\left\{\frac{mz-\alpha^2x}{\alpha|z-mx|}\right\}$$

$$=-\frac{\epsilon}{\pi\sqrt{(m^2-\alpha^2)}}\cosh^{-1}\left\{\frac{m-\alpha\sigma}{|\alpha-m\sigma|}\right\}. \tag{9}$$

Again we see that $\partial\phi/\partial z=0$ when $\sigma=\pm1$. It may be verified in this case that $\partial\phi/\partial z=0$ over the whole surface of the Mach cone from O. There is now a singularity in the value of

$\partial\phi/\partial z$ at the leading edge of the aerofoil, given by $\sigma = \alpha/m$. Physically this is not surprising, as with the sweep back beyond the Mach line it is no longer possible for a shock wave to form at the leading edge, and the discontinuous change in $\partial\phi/\partial y$ there may be expected to lead to an infinity in $\partial\phi/\partial z$. In the neighbourhood of the leading edge we must accordingly expect disagreement between the linearized theory and the actual course of events. How far this disagreement will vitiate the results for the drag over the whole aerofoil surface must be a matter for experiment, or for more exact calculations. Such local singularities have, of course, appeared before in the theory, e.g. in the flow round the edge of the aerofoil in the case of lift, and, as in that case, their effect on the aerofoil forces as a whole may be small.

Drag on double-wedge aerofoil

We are now in a position to study the variation of the drag on a double-wedge aerofoil with various angles of sweep-back. We suppose the cross-section of the aerofoil parallel to the stream to remain unaltered as the aerofoil is swept back, with chord c, and wedge semi-angle ϵ. The plan area of the aerofoil is also assumed constant, say $S = 2bc$, where $2b$ is the span with the leading edge normal to the stream. The wing tips are assumed parallel to the stream throughout (figure 11).

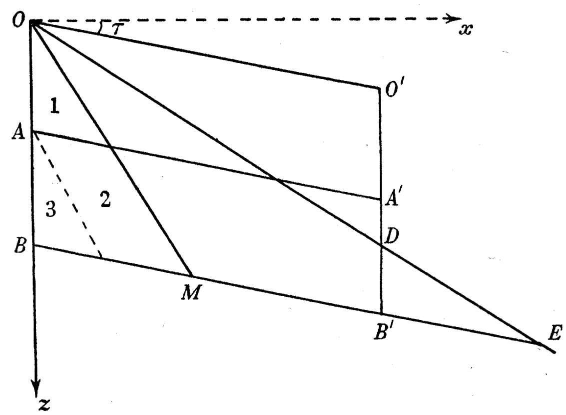

Figure 11

A slight extension of the result obtained earlier (Part I) regarding the absence of induced drag on a finite aerofoil is useful. We consider a semi-infinite swept-back aerofoil. Outside the Mach cone from O the pressure has its two-dimensional value, $\pm\Delta p_0$ say, on the front and back half of the aerofoil respectively. Within the triangle BOM there is a certain pressure defect as compared with the two-dimensional value. In the regions 1, 2, 3, as shown in figure 11, the pressure defect has the form

$$\left.\begin{array}{ll} \text{region } 1 & \Delta p_0 f(\lambda_1), \\ 2 & \Delta p_0 f(\lambda_1), \\ 3 & \Delta p_0\{f(\lambda_1) - 2f(\lambda_2)\}. \end{array}\right\} \qquad (10)$$

Here λ_1, λ_2 denote the angles between the radii from O and A respectively to the point of measurement and the stream direction, and the form of $f(\lambda)$ may be obtained from (8). It is only important to note that $f(\lambda) = 0$, $\lambda \geqslant \mu$.

The drag produced by this pressure defect over triangle BOM can now easily be calculated. It is

$$\Delta D = \epsilon \Delta p_0 \left\{ \int_1 f(\lambda_1)\, dS - \int_2 f(\lambda_1)\, dS + 2\int_3 f(\lambda_2)\, dS - \int_3 f(\lambda_1)\, dS \right\}$$
$$= \epsilon \Delta p_0 \left\{ 4\int_1 f(\lambda_1)\, dS - \int_\Delta f(\lambda_1)\, dS \right\} = 0. \tag{11}$$

Thus the drag defect over triangle BOM is zero. The result has been proved for a double-wedge aerofoil, but can easily be extended to any symmetrical swept-back aerofoil with constant cross-section parallel to the stream.

We shall now consider the evaluation of the drag on the finite wing considered above, as the angle of sweep-back τ is increased from zero.

(i) At normal incidence ($\tau = 0$) the drag force is given by

$$D = \rho U^2 \epsilon^2 S \frac{1}{\alpha}.$$

(ii) As τ is increased, by the result proved above, the induced drag due to the ends O, O' will remain zero until the Mach line from O cuts the other edge of the aerofoil between $A'B'$. This will happen when $\tan\tau = \alpha - 1/A$ (we assume, of course, that $A = (2b/c) > 1/\alpha$). Up to this critical value of τ the drag is given by

$$D = \rho U^2 \epsilon^2 S \frac{1}{\sqrt{(\alpha^2 - m^2)}}, \quad m = \tan\tau. \tag{12}$$

(iii) The third phase is reached when $\tau > \tan^{-1}(\alpha - 1/A)$, but $\tau < \tan^{-1}\alpha$. When the Mach line from O, say ODE (figure 2), has not yet reached A', the drag is easily calculated as

$$D = \rho u^2 \epsilon^2 S \frac{1}{\sqrt{(\alpha^2 - m^2)}} \left\{ 1 - \frac{1}{S}\int_\delta f(\lambda_1)\, dS \right\},$$

where δ denotes triangle $B'DE$, and $f(\lambda_1)$ is obtained from (7), so that if

$$\sigma_1 = \frac{\tan\lambda_1}{\tan\mu}, \quad f(\lambda_1) = \frac{2}{\pi}\sin^{-1}\sqrt{\left\{ \frac{(1-\sigma_1)(\alpha+m)}{2\alpha\left(1 - \frac{m}{\alpha}\sigma_1\right)} \right\}}.$$

Similarly, if the Mach line from O cuts the edge $O'B'$ between O' and A',

$$D = \rho u^2 \epsilon^2 S \frac{1}{\sqrt{(\alpha^2 - m^2)}} \left\{ 1 - \frac{1}{S}\int_\delta f(\lambda_1)\, dS + \frac{4}{S}\int_{\delta'} f(\lambda_1)\, dS \right\},$$

where δ, δ' denote the triangles cut off beyond the edge $O'B'$ between the Mach line from O and the lines BB', AA' respectively.

(iv) $\tau = \tan\alpha$. The leading edge is now along the Mach line from O. The value of $\partial\phi/\partial z$ has not yet been given in this case. It can be found by taking the limit of (7) or (9) as $m \to \alpha$. Alternatively, it can be calculated directly from (6) substituting $m = \alpha$. In place of (7) we find for the single-wedge semi-infinite aerofoil

$$\frac{\partial\phi}{\partial z} = -\frac{\epsilon}{\pi\alpha}\sqrt{\left\{\frac{1+\sigma}{1-\sigma}\right\}}. \tag{13}$$

As we might expect this vanishes at $\sigma = -1$ and has an infinity as $\sigma \to 1$.

For the finite double-wedge aerofoil we have on the surface of the wing, outside the Mach cone from O,

$$\Delta p = \frac{\rho U^2 \epsilon}{\pi \alpha} \sqrt{\left\{\frac{1+\sigma_1}{1-\sigma_1}\right\}} \quad \text{on the front half of the wing}$$

$$= -\frac{\rho U^2 \epsilon}{\pi \alpha} \left\{2\sqrt{\left\{\frac{1+\sigma_2}{1-\sigma_1}\right\}} - \sqrt{\left\{\frac{1+\sigma_1}{1-\sigma_1}\right\}}\right\} \quad \text{on the back half of the wing.} \tag{14}$$

Inside the Mach cone from O' there is a pressure defect for which the expression may immediately be written down. However, by the general result (11), this pressure defect will not affect the integrated drag.

The analytical result for the wave drag in this case can be derived from (14) by straightforward integration, but is somewhat elaborate and will not be quoted here. A numerical case is included below.

(v) $\tau > \tan \alpha$. The sweep-back is now beyond the Mach line, and the pressures on the wing must be calculated from (9). The pressure defect in the Mach cone from O' will again make no contribution to the integrated drag. The expression for the total drag force is somewhat complicated but will be given here for reference. If we define parameters P, Q, such that

$$P = \frac{m}{\alpha}, \quad Q = \frac{4b}{c} \frac{(m^2-\alpha^2)}{\alpha},$$

then the drag force D is given by

$$D = \frac{\rho U^2 \epsilon^2 S}{2\pi \sqrt{(m^2-\alpha^2)}} I, \tag{15}$$

where

$$I = \frac{Q}{\sqrt{(P^2-1)}} \left[2\cosh^{-1}\left(P+\frac{P^2-1}{Q}\right) - \tfrac{3}{2}\cosh^{-1}P - \tfrac{1}{2}\cosh^{-1}\left\{P+\frac{2(P^2-1)}{Q}\right\}\right]$$
$$+\frac{2}{Q}[\sqrt{\{(P+\tfrac{1}{2}Q)^2-1\}} - \sqrt{\{(P+Q)^2-1\}}]$$
$$+\left(\frac{2P}{Q}+4\right)\cosh^{-1}(P+Q) - 2\left(\frac{P}{Q}+1\right)\cosh^{-1}(P+\tfrac{1}{2}Q).$$

Numerical values of drag

As a numerical example the drag has been worked out for a double-wedge aerofoil for various aspect ratios in a stream with undisturbed Mach number equal to $\sqrt{2}$. The dimensionless coefficient $C = \alpha D/\rho u^2 \epsilon^2 S$ is plotted against angle of sweep-back of the leading edge in figure 12. For swept-back aerofoils the aspect ratio is, as usual, taken as the breadth normal to the stream divided by the chord. The coefficient C, as defined, starts with the value 1·0 at zero sweep-back. It rises initially as the sweep-back is increased, reaching a cusp-shaped maximum value when the leading edge lies along the Mach cone from the leading corner. Thereafter the value decreases quite steeply, but, for example, for $A = 4$ it is still 0·93 when the angle of sweep-back $\tau = 60°$. Thus it is clear that for the conditions considered of constant-aspect ratio and constant cross-section parallel to the stream sweep-back is only valuable in decreasing wave drag when it is very pronounced.

The general effect of sweep-back in diminishing wave drag is perhaps made clearest by the hydrodynamic analogue of the long-gravity waves produced by a vertical barrier in an infinite sheet of water, when the two sides of the barrier are moved apart with a given (infinitesimal) velocity. The wave drag in the aerofoil problem is proportional to the work done by the barrier in the corresponding hydrodynamic problem. Initially, when such a barrier is jerked into motion the water takes on a certain difference of level between the two sides, but as the motion proceeds the difference diminishes, and finally disappears. Broadly, the analogy to sweep-back of an aerofoil is for such a barrier to start with only a small breadth, so that little work is needed to produce the motion, and to extend in length only as the wave falls in level, so that the total work required to maintain a barrier of given length in transverse motion at a given velocity for a given time is reduced.

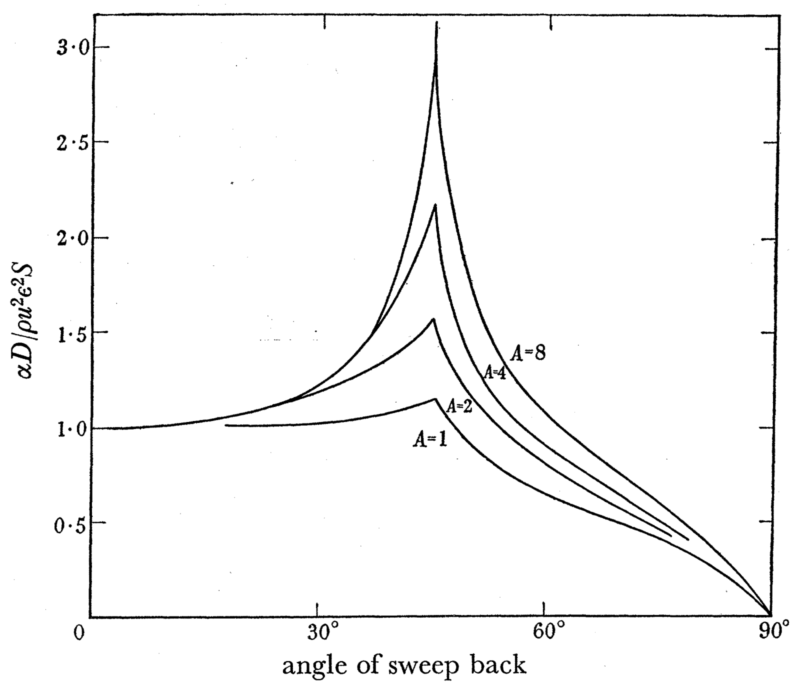

FIGURE 12. Drag on double-wedge aerofoil at zero incidence for various aspect ratios. Incident Mach number $= \sqrt{2}$.

The numerical results show that the saving in work done is small, when the particular boundary conditions for the aerofoil problem are introduced, unless the rate of expansion in length of the barrier is much less than the velocity of wave propagation. It may well be, in the aerofoil problem, that, in practice, the other effects of sweep-back, such as the prevention of the formation of shock waves at the aerofoil surface, are much the most important.

WING TIPS NOT PARALLEL TO STREAM (PROBLEMS OF INCIDENCE)

The general 'Fourier integral' method used in the treatment of symmetrical aerofoils at zero incidence is not applicable where lift is present, and $\partial\phi/\partial y$ no longer vanishes over the plane $y = 0$ except on the aerofoil's surface. It is, however, possible to develop various methods to deal with such cases. The most direct method, where one of the wing edges lies at an angle τ to the stream, say, as Oz' in figure 13, is simply to rotate our axes, and work with the co-ordinates (x', y, z'). In terms of these co-ordinates the fundamental equation (1) for the disturbance velocity potential, ϕ, becomes

$$(1-M^2\sin^2\tau)\frac{\partial^2\phi}{\partial x'^2}+\frac{\partial^2\phi}{\partial y^2}=(M^2\cos^2\tau-1)\frac{\partial^2\phi}{\partial z'^2}+2M^2\sin\tau\cos\tau\frac{\partial^2\phi}{\partial x'\,\partial z'}. \tag{16}$$

If Oz' lies within the Mach cone from O, then $M^2\sin^2\tau<1$. Equally, $M^2\cos^2\tau>1$ implies that Ox' lies outside the Mach cone from O. For definiteness we suppose in the first place that these two conditions hold. Then introducing

$$\xi=\frac{x'}{\surd(1-M^2\sin^2\tau)}, \tag{17}$$

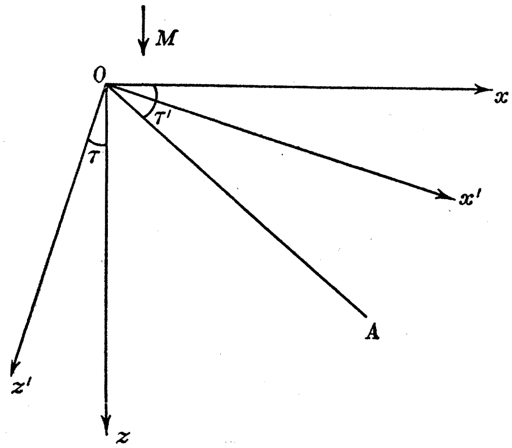

FIGURE 13

(16) can be put in the form
$$\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial y^2}-2k\frac{\partial^2\phi}{\partial\xi\,\partial z'}=\beta^2\frac{\partial^2\phi}{\partial z'^2}, \tag{18}$$

where
$$k=\frac{M^2\sin\tau\cos\tau}{\surd(1-M^2\sin^2\tau)}\quad:\quad \beta^2=(M^2\cos^2\tau-1).$$

If $\bar\phi'$ denotes the Laplace transform of ϕ with respect to z', and we write $\bar\phi'=e^{kp\xi}u$, then, if it can be assumed that $\phi=\left(\frac{\partial\phi}{\partial z'}\right)=0$ over the plane $z'=0$, u will satisfy the equation

$$\frac{\partial^2u}{\partial\xi^2}+\frac{\partial^2u}{\partial y^2}=\alpha'^2p^2u, \tag{19}$$

where
$$\beta^2+k^2=\alpha'^2=\frac{M^2-1}{1-M^2\sin^2\tau}.$$

This equation is of a type considered in Part I. To illustrate its application we shall first deal with a case the solution of which we already know from (7) and (9), viz. where the aerofoil is symmetrical, with plan $z'Ox'$, and has everywhere a slope ϵ in the direction of the stream. The boundary condition is then that $\partial\phi/\partial y$ takes the values $+\epsilon$, $-\epsilon$ respectively on the upper and lower surfaces of the aerofoil plan, so we find

$$\left(\frac{\partial u}{\partial y}\right)\to\pm\frac{\epsilon}{p}e^{-kp\xi}\quad\text{as } y\to\pm0,\quad \xi\geqslant0. \tag{20}$$

We must also assume $M^2\cos^2\tau > 1$, otherwise the assumption of zero disturbance on the plane $z' = 0$ would be unjustified. Using the Green's function method developed earlier we immediately have that at the point (ξ_0, y_0)

$$\bar{\phi}' = -\frac{\epsilon}{\pi p} e^{-kp\xi_0} \int_0^\infty d\xi \int_0^\infty \frac{\exp\{-p[k\xi + \alpha'\sqrt{\{(\xi-\xi_0)^2 + y_0^2 + \eta^2\}}]\}}{\sqrt{\{(\xi-\xi_0)^2 + y_0^2 + \eta^2\}}} d\eta. \tag{21}$$

We shall consider only points on the aerofoil, for which $\xi_0 > 0, y_0 = 0$. We find then

$$\bar{\phi}' = -\frac{\epsilon}{\pi p} \int_0^\infty d\xi \int_0^\infty \frac{\exp\{-p[k(\xi-\xi_0) + \alpha'\sqrt{\{(\xi-\xi_0)^2 + \eta^2\}}]\}}{\sqrt{\{(\xi-\xi_0)^2 + \eta^2\}}} d\eta. \tag{22}$$

This integral is of a similar type to some considered earlier, and admits of a simple geometrical interpretation, if we introduce polar co-ordinates (R, θ) in the (ξ, η) plane with pole at the point $(\xi_0, 0)$. When we take the inverse transform with respect to z' of (22), we find simply

$$\phi(x_0', z_0') = -\frac{\epsilon}{\pi} \iint_S dR\, d\theta, \tag{23}$$

where S denotes the area between the ellipse

$$R = z_0'/(\alpha' + k\cos\theta)$$

and the positive $\xi\eta$ axes (figure 14). If θ_1 denotes the angle shown in figure 14, such that

$$\cos\theta_1 = -\frac{\alpha'\xi_0}{z_0' + k\xi_0},$$

then we have
$$\phi(x_0', z_0') = -\frac{\epsilon}{\pi}\left[z_0' \int_0^{\theta_1} \frac{d\theta}{\alpha' + k\cos\theta} + \xi_0 \int_0^{\pi-\theta_1} \sec\theta\, d\theta\right], \tag{24}$$

so that
$$\left.\begin{aligned} \frac{\partial\phi}{\partial z_0'} &= \frac{\epsilon}{\pi}\int_0^{\theta_1} \frac{d\theta}{\alpha' + k\cos\theta} = -\frac{\epsilon}{\pi\beta}\cos^{-1}\left(\frac{\alpha'\cos\theta_1 + k}{\alpha' + k\cos\theta_1}\right), \\ \frac{\partial\phi}{\partial\xi_0} &= -\frac{\epsilon}{\pi}\int_0^{\pi-\theta_1} \sec\theta\, d\theta = -\frac{\epsilon}{\pi}\cosh^{-1}\{\sec(\pi-\theta_1)\}. \end{aligned}\right\} \tag{25}$$

After some transformations we thus find that at the point (x_0, z_0) on the wing, within the Mach cone from the leading corner,

$$\left.\begin{aligned} \frac{\partial\phi}{\partial z'} &= -\frac{\epsilon}{\pi\sqrt{(M^2\cos^2\tau - 1)}}\cos^{-1}\left(\frac{mz_0 - \alpha^2 x_0}{\alpha(z_0 - mx_0)}\right), \quad \text{where } m = \tan\tau, \\ \frac{\partial\phi}{\partial x'} &= -\frac{\epsilon}{\pi\sqrt{(1 - M^2\sin^2\tau)}}\cosh^{-1}\left(\frac{z_0 + m\alpha^2 x_0}{\alpha(mz_0 + x_0)}\right). \end{aligned}\right\} \tag{26}$$

The expression for
$$\frac{\partial\phi}{\partial z} = \cos\tau\frac{\partial\phi}{\partial z'} + \sin\tau\frac{\partial\phi}{\partial x'}$$

may then be formed, and it will be seen that it is identical with that which would have been obtained simply by superposing (7) and (9) with the appropriate value for m in each case, viz.

$$\frac{\partial\phi}{\partial z} = -\frac{\epsilon}{\pi\sqrt{(\alpha^2 - m^2)}}\cos^{-1}\frac{mz_0 - \alpha^2 x_0}{\alpha(z_0 - mx_0)} - \frac{\epsilon}{\pi\sqrt{(1/m^{-2} - \alpha^2)}}\cosh^{-1}\left\{\frac{z_0 + m\alpha^2 x_0}{\alpha(mz_0 + x_0)}\right\}. \tag{27}$$

Flat-plate aerofoil at incidence

The case of the flat-plate aerofoil with plan $z'Ox'$ at incidence ϵ can be worked out by exactly similar methods. We assume again that $M\sin\tau<1$, $M\cos\tau>1$. With the same notation as previously the boundary condition for $\partial\phi/\partial y$ then gives

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = -\frac{\epsilon}{p}e^{-kp\xi}, \quad \xi \geqslant 0. \tag{28}$$

Using the appropriate Green's function we have, at the point (ξ_0, y_0),

$$\bar{\phi}' = \frac{\epsilon}{\pi p}e^{-kp\xi_0}\int_0^\infty d\xi \int_0^{2\sqrt{(\xi\xi_0)}} \frac{\exp[-p\{k\xi+\alpha'\sqrt{[(\xi-\xi_0)^2+y_0^2+\eta^2]}\}]}{\sqrt{\{(\xi-\xi_0)^2+y_0^2+\eta^2\}}}\,d\eta. \tag{29}$$

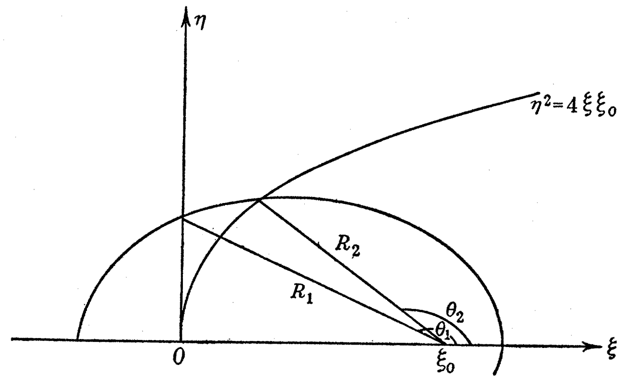

Figure 14

We shall again consider only points on the aerofoil, $y=0$, $\xi_0>0$. Introducing co-ordinates (R,θ) as before we find

$$\phi(P_0) = \phi(\xi_0, z_0') = \frac{\epsilon}{\pi}\iint_S dR\,d\theta, \tag{30}$$

where now S denotes the area in the $\xi\eta$ plane between the ξ-axis, the ellipse

$$R = \frac{z_0'}{(\alpha'+k\cos\theta)},$$

and the parabola $\eta^2 = 4\xi\xi_0$. θ_2, the angular co-ordinate of the intersection of the ellipse and parabola, is given by

$$\cos\theta_2 = \frac{z_0'-2\xi_0\alpha'}{2\xi_0 k+z_0'}. \tag{31}$$

It may be easily verified that $\theta_2=\pi$, if

$$\frac{z_0'}{x_0'} = \tan[\tfrac{1}{2}\pi-(\tau+\mu)],$$

i.e. if the point P_0 lies at the Mach angle from the leading corner; for smaller z_0'/x_0' the ellipse lies entirely within the parabola. For points within the Mach cone from the leading corner

$$\phi = \frac{\epsilon}{\pi}\left\{z_0'\int_0^{\theta_2}\frac{d\theta}{\alpha'+k\cos\theta}+2\xi_0\int_{\theta_2}^{\pi}\frac{d\theta}{1-\cos\theta}\right\}, \tag{32}$$

so that
$$\left.\begin{aligned}\frac{\partial\phi}{\partial z_0'} &= \frac{\epsilon}{\pi}\int_0^{\theta_2}\frac{d\theta}{\alpha'+k\cos\theta}+\left(\frac{z_0'}{\alpha'+k\cos\theta_2}-\frac{2\xi_0}{1-\cos\theta_2}\right)\frac{d\theta_1}{dz_0'}=\frac{\epsilon}{\pi}\int_0^{\theta_2}\frac{d\theta}{\alpha'+k\cos\theta},\\ \frac{\partial\phi}{\partial\xi_0} &= \frac{2\epsilon}{\pi}\int_{\theta_2}^{\pi}\frac{d\theta}{1-\cos\theta}.\end{aligned}\right\}\tag{33}$$

Hence we find

$$\frac{\partial\phi}{\partial z_0'}=\frac{2\epsilon}{\pi\sqrt{(\alpha'^2-k^2)}}\sin^{-1}\sqrt{\left\{\frac{\xi_0}{z_0'}(\alpha'-k)\right\}}=\frac{2\epsilon}{\pi\sqrt{(M^2\cos^2\tau-1)}}\sin^{-1}\sqrt{\left\{\frac{x_0'}{z_0'}\frac{(\alpha-\tan\tau)}{(1+\alpha\tan\tau)}\right\}}.$$

If, as previously, we denote the angle $P_0\hat{O}z$ by λ, and write $\sigma=\tan\lambda/\tan\mu$, then $\partial\phi/\partial z'$ can be written in the form

$$\frac{\partial\phi}{\partial z'}=\frac{2\epsilon}{\pi\sqrt{(M^2\cos^2\tau-1)}}\sin^{-1}\sqrt{\left(\frac{\alpha\tan\tau+\sigma}{\alpha-\sigma\tan\tau}\frac{\alpha-\tan\tau}{1+\alpha\tan\tau}\right)}.\tag{34}$$

As we should expect $(\partial\phi/\partial z')$ vanishes along Oz' at the edge of the aerofoil where

$$\sigma=-\alpha\tan\tau,$$

and reaches its two-dimensional value when $\sigma=1$. However, the whole contribution to $\partial\phi/\partial z$ does not come from $\partial\phi/\partial z'$; we must also consider the value of $\partial\phi/\partial\xi_0$, which yields

$$\begin{aligned}\frac{\partial\phi}{\partial x_0'} &= \frac{2\epsilon}{\pi\sqrt{(1-M^2\sin^2\tau)}}\sqrt{\left\{\frac{z_0'-\xi_0(\alpha'-k)}{\xi_0(\alpha'+k)}\right\}}\\ &= \frac{2\epsilon}{\pi\cos^2\tau\,(1+\alpha\tan\tau)}\sqrt{\left\{\frac{z-\alpha x}{(\alpha+\tan\tau)\,(x+z\tan\tau)}\right\}}\\ &= \frac{2\epsilon}{\pi\cos^2\tau\,(1+\alpha\tan\tau)}\sqrt{\left\{\frac{1-\sigma}{(\alpha+\tan\tau)\left(\frac{\sigma}{\alpha}+\tan\tau\right)}\right\}}.\end{aligned}\tag{35}$$

It may incidentally be verified as $\tau\to 0$ that (34) and (35) reduce to the values given earlier (Part I),

$$\frac{\partial\phi}{\partial z}=\frac{2\epsilon}{\pi\alpha}\sin^{-1}\sqrt{\sigma},\quad \frac{\partial\phi}{\partial x}=\frac{2\epsilon}{\pi}\sqrt{\left(\frac{1}{\sigma}-1\right)}.$$

Compounding the results (34) and (35) we finally obtain for the general case

$$\begin{aligned}\frac{\partial\phi}{\partial z} &= \frac{2\epsilon\cos\tau}{\pi\sqrt{(M^2\cos^2\tau-1)}}\sin^{-1}\sqrt{\left\{\frac{(\alpha\tan\tau+\sigma)\,(\alpha-\tan\tau)}{(\alpha-\sigma\tan\tau)\,(1+\alpha\tan\tau)}\right\}}\\ &\quad+\frac{2\epsilon}{\pi\,(\alpha+\cot\tau)}\sqrt{\left\{\frac{(1-\sigma)}{(\alpha\cos\tau+\sin\tau)\left(\frac{\sigma}{\alpha}\cos\tau+\sin\tau\right)}\right\}}.\end{aligned}\tag{36}$$

More general case—flat-plate aerofoil at incidence

The more general case of a flat-plate aerofoil with plan $z'OA$ (figure 13), at incidence ϵ in the stream M, can be treated by an obvious extension of the methods used above, provided at least one of the edges Oz', OA lies outside the Mach cone from O. We suppose that, in fact, Oz' lies within and OA without the Mach cone. Then, with the notation used previously, we find for u the boundary condition

$$\left.\begin{aligned}\left(\frac{\partial u}{\partial y}\right)_{y=0} &= -\frac{\epsilon}{p}e^{-kp\xi-p\xi\tan(\tau'-\tau)\sqrt{(1-M^2\sin^2\tau)}} \quad (\xi>0)\\ &= -\frac{\epsilon}{p}e^{-m'p\xi}, \quad \text{say.}\end{aligned}\right\} \tag{37}$$

Hence we can deduce, for the transform $\bar{\phi}'$ at a point P_0 of the aerofoil,

$$\bar{\phi}'(P_0) = \frac{\epsilon}{p\pi}\int_0^\infty\int_0^{2\sqrt{(\xi\xi_0)}} \frac{\exp\left[-p\{m'\xi-k\xi_0+\alpha'\sqrt{[(\xi-\xi_0)^2+\eta^2]}\}\right]}{\sqrt{\{(\xi-\xi_0)^2+\eta^2\}}}\,d\xi\,d\eta. \tag{38}$$

Inverting the transform we find, with (R, θ) as before,

$$\phi(P_0) = \frac{\epsilon}{\pi}\iint_S dR\,d\theta,$$

where S now denotes the area between the ξ-axis, the ellipse

$$z_0' = (m'-k)\,\xi_0 + (\alpha'+m'\cos\theta)\,R$$

and the parabola $$\eta^2 = 4\xi\xi_0.$$

Hence we have immediately, for points on the aerofoil within the Mach cone from O,

$$\phi(P_0) = \frac{\epsilon}{\pi}\left[\{z_0'-(m'-k)\,\xi_0\}\int_0^{\theta_1}\frac{d\theta}{\alpha'+m'\cos\theta} + 2\xi_0\int_{\theta_1}^{\pi}\frac{d\theta}{1-\cos\theta}\right], \tag{39}$$

where now $$\cos\theta_1 = \frac{z_0'-(m'-k)\,\xi_0-2\alpha'\xi_0}{2\xi_0 k+z_0'-(m'-k)\,\xi_0}.$$

The reduction of the expression, which can hence be obtained for $\partial\phi/\partial z$, is somewhat tedious, and the simplicity of the final result suggests that there may be some more elegant method of derivation. The formulae

$$(\alpha'^2-m'^2) = \sec^2(\tau'-\tau)\,(M^2\cos^2\tau'-1),$$

$$(\alpha'-m') = \sec(\tau'-\tau)\cos\tau\cos\tau'\,(1-\alpha\tan\tau)\,(\alpha-\tan\tau'),$$

and $$\frac{\alpha'+m'}{\alpha'-m'} = \frac{(1+\alpha\tan\tau)\,(\alpha+\tan\tau')}{(\alpha-\tan\tau')\,(1-\alpha\tan\tau)}$$

are helpful in effecting the straightforward reduction which finally leads to

$$\begin{aligned}\frac{\partial\phi}{\partial z} = \frac{2\epsilon}{\pi}\frac{\cos\tau'}{\sqrt{(M^2\cos^2\tau'-1)}}\sin^{-1}\sqrt{\left\{\left(\frac{\sigma+\alpha\tan\tau}{1+\alpha\tan\tau}\right)\left(\frac{\alpha-\tan\tau'}{\alpha-\sigma\tan\tau'}\right)\right\}}\\ +\frac{2\epsilon}{\pi}\frac{1}{(\alpha+\cot\tau)}\sqrt{\left\{\frac{\alpha(1+\tan\tau\tan\tau')\,(1-\sigma)}{(\sigma+\alpha\tan\tau)(\alpha+\tan\tau')}\right\}},\end{aligned} \tag{40}$$

where, as usual, $\sigma = \alpha x/z$. The result holds for either positive or negative τ and τ', subject only to the restrictions stated in the first paragraph of this section.

The case when $\tau' = 0$ leads to the particularly simple result

$$\frac{\partial\phi}{\partial z} = \frac{2\epsilon}{\pi\alpha}\sin^{-1}\sqrt{\left(\frac{\sigma+\alpha\tan\tau}{1+\alpha\tan\tau}\right)} + \frac{2\epsilon}{\pi(\alpha+\cot\tau)}\sqrt{\left(\frac{(1-\sigma)}{\sigma+\alpha\tan\tau}\right)}. \tag{41}$$

So far we have restricted our discussion, where lift is involved, to cases where only one edge of the aerofoil is swept back 'beyond the Mach angle'. An instance of the type of problem not considered is an aerofoil of arrow-head plan pointing upstream, and with the semi-angle of the arrow less than the Mach angle, so that the arrow-head lies entirely within the Mach cone from its leading point. A direct solution of the boundary-value problem is difficult in this case, and to the author's knowledge none has been given.* However, the formal difficulties in the way of a direct solution are strongly reminiscent of those encountered in the analysis of the extended rectangular plate at incidence for which the treatment was given in Part I, using the Green's function method, and similar to Schwarzschild's (1902) method in treating diffraction at a slit. The physical analogy becomes clear if we consider the arrowhead aerofoil as the limiting case of a trapezoidal aerofoil (figure 15), when the breadth $2b$ of the leading edge tends to zero.

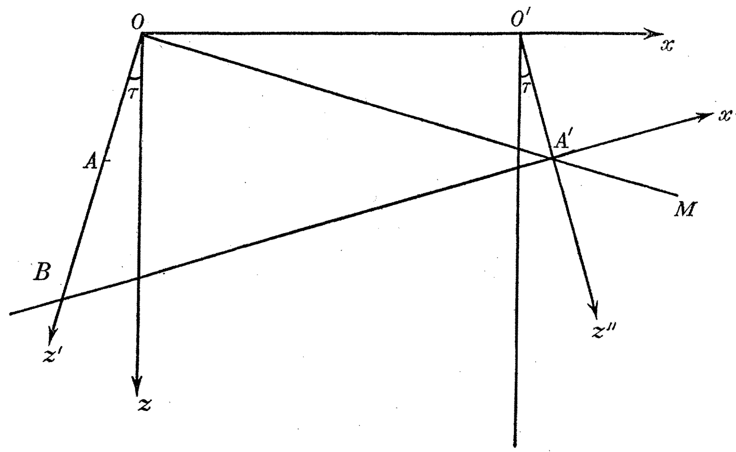

FIGURE 15

In the last section we have found the velocity potential for flow against a semi-infinite flat plate aerofoil with plan $z'Ox$ (figure 13) at incidence. Using the same analogy as in Part I this can be treated as the superposition of a plane wave, giving the two-dimensional value for the pressure and other quantities over the aerofoil, together with a circular, or more properly cone, wave C_1 say, spreading from the corner O. From the results of the previous section it is easily deduced that the velocity potential of this cone wave is, in fact,

$$C_1 = -\frac{2\epsilon}{\pi}\frac{z}{\alpha}\left[\cos^{-1}\sqrt{\left(\frac{\sigma+\alpha\tan\tau}{1+\alpha\tan\tau}\right)}-\frac{\sqrt{(1-\sigma)}\sqrt{(\sigma+\alpha\tan\tau)}}{(1+\alpha\tan\tau)}\right]. \tag{42}$$

The 'trapezoidal problem' of flow against the plate with plan $z'OO'z''$ will be satisfactorily solved by superposing the plane wave, confined to its area, together with the two cone waves C_1, C_1', until these cone waves run off the aerofoil edges at $A'A$ respectively. Behind these points, although our solution still satisfies the differential equation, and $\partial\phi/\partial y$ has the correct value over the aerofoil the solution is invalid on account of the discontinuity in ϕ which appears over the triangular areas of which $MA'Z''$ is typical. The validity of the solution is restored by superposing two further waves, in this case no longer of the 'cone-field'

* Since this was written the author has been informed by Ward that he and Robinson have obtained the solution in a closed form.

type beginning at the points A, A' and radiating from one edge over to the other. The process has to be repeated at B, B' and so on indefinitely until the end of the aerofoil is reached. Physically it is clear that the process will be a convergent one, consisting as it does of the diminishing oscillations of excess pressure between the upper and undersides of the aerofoil. A water wave analogy in this connexion has already been considered in Part I.

The convergence is important when we come to the limiting case of the 'arrowhead' aerofoil, for then the complete infinite set of superposed waves is necessary for the solution. A numerical example of this case is discussed below, from which it is clear that in certain circumstances the convergence is quite quick. Apart from the requirement to superpose the complete set of waves the arrowhead case is really simpler to treat than the trapezoidal one, for each separate wave is then of the cone-field type, and accordingly requires much less calculation. We shall, however, begin by considering the trapezoidal case.

Referred to the axes $A'x''$, $A'y$, $A'z''$ (figure 15) the differential equation of the wave motion becomes

$$\frac{\partial^2\phi}{\partial\xi''^2}+\frac{\partial^2\phi}{\partial y^2}+2k\frac{\partial^2\phi}{\partial\xi''\partial z''}=\beta^2\frac{\partial^2\phi}{\partial z''^2}, \tag{43}$$

where k, β are as already defined (18), and $\xi''=\dfrac{x''}{\sqrt{(1-M^2\sin^2\tau)}}$. If $\bar{\phi}''$ denotes the Laplace transform with respect to z'', for a wave beginning at $z''=0$, then

$$\frac{\partial^2\bar{\phi}''}{\partial\xi''^2}+\frac{\partial^2\bar{\phi}''}{\partial y^2}+2kp\frac{\partial\bar{\phi}''}{\partial\xi''}=\beta^2p^2\bar{\phi}'', \tag{44}$$

and if we define
$$u=e^{kp\xi''}\bar{\phi}'',$$

$$\frac{\partial^2u}{\partial\xi''^2}+\frac{\partial^2u}{\partial y^2}=\alpha'^2p^2u. \tag{45}$$

If we denote by $F(\xi'')$ the z'' transform of C_1, for $x''>0$, then in order to annul the potential discontinuity introduced in the $\Delta z''A'M$ by C_1, we must find u satisfying (45) with the boundary conditions that $u=e^{kp\xi''}F(\xi'')$ along the positive x''-axis, and $\partial u/\partial y=0$ along the negative x''-axis. This problem has already been treated (Part I), and the solution for the consequent $\bar{\phi}''$ at a point of the negative x''-axis given by $\xi''=-\vartheta$ say, is

$$\bar{\phi}_2''=\frac{1}{\pi}\int_0^\infty F(\xi'')\sqrt{\left(\frac{\vartheta}{\xi''}\right)}\frac{e^{-m'p(\vartheta+\xi'')}}{(\vartheta+\xi'')}\,d\xi'', \tag{46}$$

where $m'=(\alpha'-k)$. It is convenient before inverting the transform to reintroduce x'' in place of ξ'', and to replace $F(\xi'')$ by its value

$$F(\xi'')=\int_0^\infty e^{-pz''}C_1(x'',z'')\,dz''.$$

At a point P on the aerofoil, the co-ordinates of which are $(-W,Z)$ referred to the (x'',z'') axes, we then find

$$\phi_2(-W,Z)=\frac{1}{\pi}\int_0^\infty\int_0^\infty\sqrt{\left(\frac{W}{x''}\right)}\frac{C_1(x'',z'')}{(W+x'')}\delta\{Z-z''-\cot(\mu+\tau)(W+x'')\}\,dx''\,dz''. \tag{47}$$

The contributions to the integral comes entirely from the neighbourhood of the line in the (x'', z'') plane, with equation

$$Z - W\cot(\mu+\tau) = z'' + x''\cot(\mu+\tau).$$

It is easily seen that the integral is zero unless $Z > W\cot(\mu+\tau)$, i.e. unless that point P lies within the Mach cone from A'. If this is so, then integrating with respect to z'',

$$\phi_2(-W, Z) = \frac{1}{\pi}\int_0^X \sqrt{\left(\frac{W}{x''}\right)} \frac{C_1\{x'', Z-(W+x'')\cot(\mu+\tau)\}}{(W+x'')}\,dx'', \tag{48}$$

where the upper limit X for the integral is fixed by the vanishing of C_1 outside the Mach cone from O, so that

$$Z-(W+X)\cot(\mu+\tau) = X\cot(\mu-\tau). \tag{49}$$

We have thus determined in terms of the wave C_1 the velocity potential for the wave C_2'. The formula can also be successively applied for the deduction of all the subsequent waves. Similar, but more complicated, results can be established for the velocity components, required to find the excess pressure, but the matter will not be pursued further here.

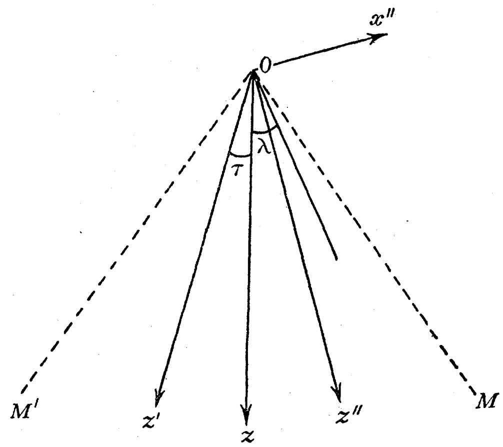

FIGURE 16

We turn rather to the arrowhead wing (figure 16) where, as pointed out above, conditions are simplified by the cone-field form of all the potentials. The solution is now found by a method of successive approximation. We start with a plane wave extended over the plan of the aerofoil, and add to this the two cone waves $C_1\,C_1'$ of the form (42) corresponding to the two edges Oz', Oz'' of the arrowhead. Then the wave C_1, for example, gives rise to a potential discontinuity in the triangle $z''OM$ which must be eliminated by an added wave, satisfying the equation of motion and continuity requirements without disturbing $\partial\phi/\partial y$ over the aerofoil. We may suppose that the velocity potential for the waves $C_1\,C_1'$ can be expressed, irrespective of sign, in the form $\frac{\epsilon z}{\alpha}f_1(\lambda)$, $\frac{\epsilon z}{\alpha}f_1'(\lambda)$, and that successive waves from the two edges are denoted by

$$\frac{\epsilon z}{\alpha}f_2(\lambda),\quad \frac{\epsilon z}{\alpha}f_2'(\lambda),\quad \frac{\epsilon z}{\alpha}f_3(\lambda),\quad \frac{\epsilon z}{\alpha}f_3'(\lambda),\quad \text{etc.}$$

Then either directly, or from (48), it is easy to deduce that, say

$$f_2'(\lambda_1) = \frac{1}{\pi}\int_\tau^\mu f_1(\lambda)\sqrt{\left(\frac{W}{x'}\right)}\frac{x'^2}{W+x'}\frac{\operatorname{cosec}^3(\lambda-\tau)\,d\lambda}{\{\cot(\mu+\tau)+\cot(\lambda-\tau)\}}, \tag{50}$$

where $x' = \dfrac{Z - W\cot(\mu+\tau)}{\cot(\mu+\tau)+\cot(\lambda-\tau)}$, and the value of f_2 is given for a point with co-ordinates $(-W, Z)$ with respect to the axes (x'', z'') (figure 16) so that $\cot(\tau-\lambda_1) = Z/W$, $\lambda_1<\tau$. (50) can be put in a more convenient form if we write

$$\sigma = \frac{\alpha x}{z} = \frac{\tan\lambda}{\tan\mu} \quad \text{for } \lambda>\tau,$$

and

$$\rho = -\frac{\alpha x}{z} = -\frac{\tan\lambda}{\tan\mu} \quad \text{for } \lambda<\tau.$$

We also denote $\tan\tau/\tan\mu = \sigma_1$, so that ρ goes from $+1$ to $-\sigma_1$ across the space $M'z''$ and σ goes from σ_1 to 1 across $z''M$. If then we put $f_1(\lambda) = g_1(\sigma)$, etc., we find from (50) after some reduction

$$g_2'(\rho) = \frac{1}{\pi}(1-\rho)^{\frac{3}{2}}\sqrt{(\rho+\sigma_1)}\int_{\sigma_1}^{1}\frac{g_1(\sigma)\,d\sigma}{(1+\sigma)^{\frac{3}{2}}(\sigma+\rho)\sqrt{(\sigma-\sigma_1)}}. \tag{51}$$

This expression in fact gives, over the aerofoil and to the left of it within the Mach cone, the 'reflected' wave corresponding to the wave $g_1(\sigma)$ coming from the edge Oz'. The complete solution can then be written, for points on the aerofoil surface

$$\phi = \frac{\epsilon z}{\alpha}\left\{1+\sum_{n=1}^{\infty}(-1)^n[g_n(\rho)+g_n'(\sigma)]\right\},$$

where 1 denotes the plane wave, and successive g_n's are to be derived from the recurrence relation (51). Also

$$g_1 = \frac{2}{\pi}\left\{\cos^{-1}\sqrt{\left(\frac{\sigma+\sigma_1}{1+\sigma_1}\right)} - \frac{\sqrt{(1-\sigma)}\sqrt{(\sigma+\sigma_1)}}{1+\sigma_1}\right\}. \tag{52}$$

The convergence of the series has not been investigated in detail. Broadly it appears that the convergence will be fairly speedy except when the arrow angle is only a very small fraction of the Mach angle. One numerical example has been considered in detail with incident Mach number $\sqrt{2}$, and arrow semi-angle 26·3°, so that $\sigma_1 = 0{\cdot}5$. The various stages in the evaluation of the potential are shown in figure 17. In this case only f_1, f_2 contributed appreciably to the result. The potential across the aerofoil is also shown for this case.

The excess pressure at any point can be derived from figure 17 by differentiation. However, the total force on the aerofoil can be found without this step. Thus, if the height of a triangular plate aerofoil is c the total lift force L is given by

$$L = 2\int_{-c\tan\tau}^{c\tan\tau}dx\int_{(c-x\cot\tau)}^{c}\rho_1 U^2\left(\frac{\partial\phi}{\partial z}\right)dz$$

$$= \frac{2\rho_1 U^2\epsilon c^2}{\alpha^2}\int_{-\sigma_1}^{\sigma_1}F(\sigma)\,d\sigma,$$

where the velocity potential for the flow is assumed expressed in the form $\phi = (\epsilon z/\alpha)F(\sigma)$, and ρ_1 now denotes the density of the gas. In the particular case considered we find $L/L_0 = 0{\cdot}65$, where L_0 is the lift force which would result from the two-dimensional excess velocity ϵ/α over the surface of the plate. It would be simple to carry out a similar investigation to that already made for drag, and to give, for example, L/L_0 as a function of the angle τ for sweep-back both less and greater than the Mach angle. However, we shall not pursue the question further at this stage.

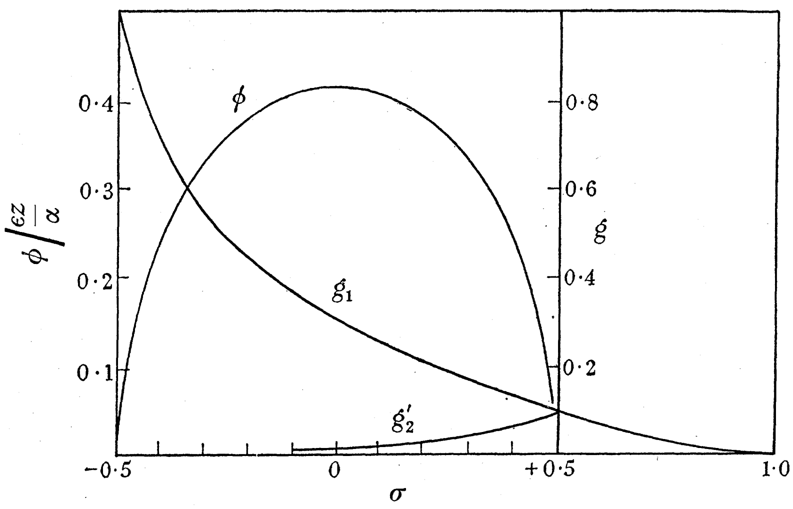

FIGURE 17. Derivation of potential across arrowhead aerofoil semi-angle 26·3°. Incident Mach number = $\sqrt{2}$.

FURTHER DEVELOPMENTS

A variety of techniques has now been presented for the solution of the boundary-value problems associated with the linearized equation for supersonic flow. By combination of the various techniques it is clear that a very wide range of aerofoil problems can be solved, with arbitrary cross-section and considerable variety in choice of plan. The only outstanding problem of mathematical interest for which a solution has not been given appears to be the case of an aerofoil at incidence, with a curved edge swept back beyond the Mach line. This problem arises, for example, in such a simple case as an elliptic plan. The hydrodynamic gravity wave analogue of the fundamental problem here is that of the waves produced by a plank moved normal to itself with a given velocity, and at the same time moved parallel to its length with a varying velocity less than the velocity of wave propagation. It appears possible that the solution may be obtained by some method of splitting the motion of the plank into a series of impulsive jerks (Heaviside unit functions), but it is not proposed to go into a detailed investigation of the matter.

It is felt that sufficient demonstration has already been given of the wide applicability of the operational treatment to linearized supersonic aerofoil theory. The advantages of the formal unity introduced, and of the generality of the problems for which solutions may be found, are clear. Before proceeding to more detailed calculations, however, it seems desirable that the practical validity of the linearized theory should be established in some of the simpler cases.

My acknowledgement and thanks are due to Professor S. Goldstein, F.R.S., at whose suggestion these investigations were started, and with whom I have had many helpful discussions during their progress.

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